

Derivation Algebras of Centerless Perfect Lie Algebras Are Complete¹

(appeared in *J. Algebra*, **285** (2005), 508–515.)

Yucan Su*, Linsheng Zhu†

* *Department of Mathematics, Shanghai Jiaotong University, Shanghai 200030, China*

† *Department of Mathematics, Changshu Institute of Technology, Jiangsu 215500, China*

Abstract: It is proved that the derivation algebra of a centerless perfect Lie algebra of arbitrary dimension over any field of arbitrary characteristic is complete and that the holomorph of a centerless perfect Lie algebra is complete if and only if its outer derivation algebra is centerless.

Key words: Derivation, complete Lie algebra, holomorph of Lie algebra

Mathematics Subject Classification (1991): 17B05

§1. Introduction

A Lie algebra is called *complete* if its center is zero, and all its derivations are inner. This definition of a complete Lie algebra was given by Jacobson in 1962 [3]. In [12], Schenkman proved a theorem, the so-called *derivation tower theorem*, that the last term of the derivation tower of a centerless Lie algebra is complete. It is also known that the holomorph of an abelian Lie algebra is complete. Suggested by the derivation tower theorem, complete Lie algebras occur naturally in the study of Lie algebras and would provide interesting objects for investigation. In 1963, Leger [6] presented some interesting examples of complete Lie algebras. However since then, the study of complete Lie algebras had become dormant for decades, partly due to the fact that the structure theory of complete Lie algebras was not well developed. In recent years, much progress has been obtained on the structure theory of complete Lie algebras in finite dimensional case (see for example, the references at the end of this paper).

¹This work is supported by the NSF of China, two grants “Excellent Young Teacher Program” and “Trans-Century Training Programme Foundation for the Talents” from Ministry of Education of China, the Doctoral Programme Foundation of Institution of Higher Education and a fund of Jiangsu Educational Committee.

It is known that a simple Lie algebra is not always complete (e.g. in some infinite dimensional cases [14]). In this case, a natural question is: *how big is the length of the derivation tower of a simple Lie algebra?*

In this paper, we answer the above question. Precisely, we prove that the derivation algebra of a *centerless perfect* Lie algebra (i.e., a Lie algebra \mathfrak{g} with zero center and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$) of arbitrary dimension over any field of arbitrary characteristic is complete, thus as a consequence, the length of the derivation tower of any simple Lie algebra is ≤ 1 .

Let \mathbb{F} be any field. First we recall the definitions of a derivation and the holomorph of a Lie algebra \mathfrak{g} over the field \mathbb{F} . A *derivation* of a Lie algebra \mathfrak{g} is an \mathbb{F} -linear transformation $d : \mathfrak{g} \rightarrow \mathfrak{g}$ such that

$$d([x, y]) = [d(x), y] + [x, d(y)] \quad \text{for } x, y \in \mathfrak{g}. \quad (1.1)$$

We denote by $\text{Der } \mathfrak{g}$ the vector space of derivations of \mathfrak{g} , which forms a Lie algebra with respect to the commutator of linear transformations, called the *derivation algebra* of \mathfrak{g} . Clearly, the space $\text{ad } \mathfrak{g} = \{\text{ad } x \mid x \in \mathfrak{g}\}$ of *inner derivations* is an ideal of $\text{Der } \mathfrak{g}$. We call $\text{Der } \mathfrak{g} / \text{ad } \mathfrak{g}$ the *outer derivation algebra* of \mathfrak{g} .

The *holomorph* $\mathfrak{h}(\mathfrak{g})$ of a Lie algebra \mathfrak{g} is the direct sum of the vector spaces $\mathfrak{h}(\mathfrak{g}) = \mathfrak{g} \oplus \text{Der } \mathfrak{g}$ with the following bracket

$$[(x, d), (y, e)] = ([x, y] + d(y) - e(x), [d, e]) \quad \text{for } x, y \in \mathfrak{g}, d, e \in \text{Der } \mathfrak{g}. \quad (1.2)$$

An element (x, d) of $\mathfrak{h}(\mathfrak{g})$ is also written as $x + d$. Obviously, \mathfrak{g} is an ideal of $\mathfrak{h}(\mathfrak{g})$ and $\mathfrak{h}(\mathfrak{g}) / \mathfrak{g} \cong \text{Der } \mathfrak{g}$. Thus we write

$$\mathfrak{h}(\mathfrak{g}) = \mathfrak{g} \rtimes \text{Der } \mathfrak{g}. \quad (1.3)$$

For a Lie algebra \mathfrak{g} , we denote by $C(\mathfrak{g})$ the center of \mathfrak{g} , i.e., $C(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{g}] = 0\}$.

The main result of this paper is the following

Theorem 1.1. *Let \mathfrak{g} be a perfect Lie algebra with zero center (i.e., $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, $C(\mathfrak{g}) = 0$). Then we have*

(i) *The derivation algebra $\text{Der } \mathfrak{g}$ is complete.*

(ii) *The holomorph $\mathfrak{h}(\mathfrak{g})$ is complete if and only if the center of outer derivation algebra is zero, i.e., $C(\text{Der } \mathfrak{g} / \text{ad } \mathfrak{g}) = 0$.*

We shall prove Theorem 1.1 in Section 2, then we give some interesting example in Section

3.

§2. Proof of the main result

Proof of Theorem 1.1(i). Assume that $d \in C(\text{Der } \mathfrak{g})$. Then in particular we have $[d, \text{ad}_x](y) = 0$ for all $x, y \in \mathfrak{g}$. Thus $d([x, y]) = [x, d(y)]$. Hence by (1.1), $[d(x), y] = 0$ for all $x, y \in \mathfrak{g}$. Since \mathfrak{g} has zero center, we obtain $d(x) = 0$, i.e., $d = 0$. Therefore

$$C(\text{Der } \mathfrak{g}) = 0. \quad (2.1)$$

Now we prove that all derivations of the Lie algebra $\text{Der } \mathfrak{g}$ are inner. First we have

Claim 1. Let $D \in \text{Der}(\text{Der } \mathfrak{g})$. If $D(\text{ad}_\mathfrak{g}) = 0$, then $D = 0$.

Let $d \in \text{Der } \mathfrak{g}$, $x \in \mathfrak{g}$. Note from (1.1) that in the Lie algebra $\text{Der } \mathfrak{g}$, we have

$$[d, \text{ad}_x] = \text{ad}_{d(x)} \in \text{ad}_\mathfrak{g}. \quad (2.2)$$

Using this, noting that $D(d) \in \text{Der } \mathfrak{g}$ and the fact that $D(\text{ad}_\mathfrak{g}) = 0$, we have

$$\begin{aligned} \text{ad}_{D(d)(x)} &= [D(d), \text{ad}_x] \\ &= D([d, \text{ad}_x]) - [d, D(\text{ad}_x)] \\ &= D(\text{ad}_{d(x)}) \\ &= 0. \end{aligned} \quad (2.3)$$

Since \mathfrak{g} has zero center, (2.3) gives $D(d)(x) = 0$ for all $x \in \mathfrak{g}$, which means that $D(d) = 0$ as a derivation of \mathfrak{g} . But d is arbitrary, we obtain $D = 0$. This proves the claim.

Since \mathfrak{g} is perfect, for any $x \in \mathfrak{g}$, we can write x as

$$x = \sum_{i \in I} [x_i, y_i] \quad \text{for some } x_i, y_i \in \mathfrak{g}, \quad (2.4)$$

and for some finite index set I . Then

$$\text{ad}_x = \sum_{i \in I} [\text{ad}_{x_i}, \text{ad}_{y_i}]. \quad (2.5)$$

Then for any $D \in \text{Der}(\text{Der } \mathfrak{g})$, we have

$$\begin{aligned} D(\text{ad}_x) &= \sum_{i \in I} D([\text{ad}_{x_i}, \text{ad}_{y_i}]) \\ &= \sum_{i \in I} ([D(\text{ad}_{x_i}), \text{ad}_{y_i}] + [\text{ad}_{x_i}, D(\text{ad}_{y_i})]). \end{aligned} \quad (2.6)$$

Let $d_i = D(\text{ad}_{x_i})$, $e_i = D(\text{ad}_{y_i}) \in \text{Der } \mathfrak{g}$. Then by (2.2), we have

$$\begin{aligned} D(\text{ad}_x) &= \sum_{i \in I} (\text{ad}_{d_i(y_i)} - \text{ad}_{e_i(x_i)}) \\ &= \text{ad}_{\sum_{i \in I} (d_i(y_i) - e_i(x_i))} \in \text{ad}_{\mathfrak{g}}. \end{aligned} \quad (2.7)$$

This means that $D(\text{ad}_x) = \text{ad}_y$ for some $y \in \mathfrak{g}$. Since $C(\mathfrak{g}) = 0$, such y is unique. Thus $d : x \mapsto y$ defines a linear transformation of \mathfrak{g} such that

$$D(\text{ad}_x) = \text{ad}_{d(x)}. \quad (2.8)$$

For $x, y \in \mathfrak{g}$, we have

$$\begin{aligned} \text{ad}_{d([x,y])} &= D(\text{ad}_{[x,y]}) \\ &= D([\text{ad}_x, \text{ad}_y]) \\ &= [D(\text{ad}_x), \text{ad}_y] + [\text{ad}_x, D(\text{ad}_y)] \\ &= [\text{ad}_{d(x)}, \text{ad}_y] + [\text{ad}_x, \text{ad}_{d(y)}] \\ &= \text{ad}_{[d(x), y] + [x, d(y)]}. \end{aligned} \quad (2.9)$$

This and the fact that $C(\mathfrak{g}) = 0$ mean that $d([x, y]) = [d(x), y] + [x, d(y)]$, i.e., $d \in \text{Der } \mathfrak{g}$. Then (2.8) gives that $D(\text{ad}_x) = \text{ad}_{d(x)} = [d, \text{ad}_x]$ for all $x \in \mathfrak{g}$, i.e.,

$$(D - \text{ad}_d)(\text{ad}_{\mathfrak{g}}) = 0. \quad (2.10)$$

By Claim I, we have $D - \text{ad}_d = 0$, i.e., $D = \text{ad}_d$ is an inner derivation on $\text{Der } \mathfrak{g}$. This together with (2.1) proves Theorem 1.1(i).

Proof of Theorem 1.1(ii). “ \Leftarrow ”: First we prove the sufficiency. So suppose $C(\text{Der } \mathfrak{g}/\text{ad}_{\mathfrak{g}}) = 0$. We want to prove that $\mathfrak{h}(\mathfrak{g})$ is complete.

First we prove $C(\mathfrak{h}(\mathfrak{g})) = 0$. Suppose $h = x + d \in C(\mathfrak{h}(\mathfrak{g}))$ for some $x \in \mathfrak{g}$, $d \in \text{Der } \mathfrak{g}$. Letting $y = 0$ in (1.2), we obtain $[d, e] = 0$ for all $e \in \text{Der } \mathfrak{g}$, i.e., $d \in C(\text{Der } \mathfrak{g}) = 0$. Then $h = x \in C(\mathfrak{h}(\mathfrak{g})) \cap \mathfrak{g} \subset C(\mathfrak{g}) = 0$, i.e., $h = 0$. Thus $C(\mathfrak{h}(\mathfrak{g})) = 0$. Then as in the proof of (2.1), we have

$$C(\text{Der}(\mathfrak{h}(\mathfrak{g}))) = 0. \quad (2.11)$$

Now let $\mathcal{D} \in \text{Der}(\mathfrak{h}(\mathfrak{g}))$. For any $x \in \mathfrak{g} \subset \mathfrak{h}(\mathfrak{g})$ written in the form (2.4), since \mathfrak{g} is an ideal of $\mathfrak{h}(\mathfrak{g})$, we have

$$\mathcal{D}(x) = \sum_{i \in I} ([\mathcal{D}(x_i), y_i] + [x_i, \mathcal{D}(y_i)]) \in \mathfrak{g}. \quad (2.12)$$

Thus $d' = \mathcal{D}|_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathfrak{g}$ is a derivation of \mathfrak{g} , i.e., $d' \in \text{Der } \mathfrak{g} \subset \mathfrak{h}(\mathfrak{g})$. Let $\mathcal{D}_1 = \mathcal{D} - \text{ad}_{d'} \in \text{Der}(\mathfrak{h}(\mathfrak{g}))$. Then $\mathcal{D}_1(x) = \mathcal{D}(x) - [d', x] = \mathcal{D}(x) - d'(x) = 0$, i.e.,

$$\mathcal{D}_1|_{\mathfrak{g}} = 0. \quad (2.13)$$

For any $d \in \text{Der } \mathfrak{g} \subset \mathfrak{h}(\mathfrak{g})$, by (1.3), we can write $\mathcal{D}_1(d) \in \mathfrak{h}(\mathfrak{g})$ as

$$\mathcal{D}_1(d) = x_d + d_1 \quad \text{for some } x_d \in \mathfrak{g}, d_1 \in \text{Der } \mathfrak{g}. \quad (2.14)$$

Then in $\mathfrak{h}(\mathfrak{g})$, by (1.2), for any $y \in \mathfrak{g}$, we have

$$\begin{aligned} [x_d, y] + d_1(y) &= [x_d + d_1, y] \\ &= [\mathcal{D}_1(d), x_d] \\ &= \mathcal{D}_1([d, x_d]) - [d, \mathcal{D}_1(x_d)] \\ &= \mathcal{D}_1(d(x_d)) - [d, \mathcal{D}_1(x_d)] \\ &= 0, \end{aligned} \quad (2.15)$$

where the last equality follows from (2.13). This means that $d_1 = -\text{ad}_{x_d}$ and so $\mathcal{D}_1(d) = x_d - \text{ad}_{x_d}$. For $d, e \in \text{Der } \mathfrak{g}$, we have

$$\begin{aligned} x_{[d,e]} - \text{ad}_{x_{[d,e]}} &= \mathcal{D}_1([d, e]) \\ &= [\mathcal{D}_1(d), e] + [d, \mathcal{D}_1(e)] \\ &= (-e(x_d) + d(x_e)) + ([-\text{ad}_{x_d}, e] + [d, -\text{ad}_{x_e}]), \end{aligned} \quad (2.16)$$

where the last equality follows from (1.2) and the fact that $\mathcal{D}_1(d) = x_d - \text{ad}_{x_d}$. So

$$\text{ad}_{x_{[d,e]}} = [\text{ad}_{x_d}, e] + [d, \text{ad}_{x_e}], \quad (2.17)$$

i.e., the linear transformation $D : \text{Der } \mathfrak{g} \rightarrow \text{Der } \mathfrak{g}$ defined by

$$D(d) = \text{ad}_{x_d} \quad \text{for } d \in \text{Der } \mathfrak{g}, \quad (2.18)$$

is a derivation of $\text{Der } \mathfrak{g}$. Since $\text{Der } \mathfrak{g}$ is complete, there exists $d'' \in \text{Der } \mathfrak{g}$ such that

$$D = \text{ad}_{d''}. \quad (2.19)$$

For any $d \in \text{Der } \mathfrak{g}$, we have

$$[d'', d] = D(d) = \text{ad}_{x_d} \in \text{ad}_{\mathfrak{g}}, \quad (2.20)$$

i.e., $d'' + \text{ad}_{\mathfrak{g}} \in C(\text{Der } \mathfrak{g}/\text{ad}_{\mathfrak{g}}) = 0$. Thus $d'' \in \text{ad}_{\mathfrak{g}}$. Therefore there exists $y \in \mathfrak{g}$ such that

$$d'' = \text{ad}_y. \quad (2.21)$$

Note that $y - \text{ad}_y \in \mathfrak{h}(\mathfrak{g})$. Let $\mathcal{D}_2 = \mathcal{D}_1 - \text{ad}_{y - \text{ad}_y}$. Then for any $x \in \mathfrak{g} \subset \mathfrak{h}(\mathfrak{g})$, we have

$$\begin{aligned}\mathcal{D}_2(x) &= \mathcal{D}_1(x) - [y - \text{ad}_y, x] \\ &= -[y, x] + \text{ad}_y(x) \\ &= -[y, x] + [y, x] \\ &= 0,\end{aligned}\tag{2.22}$$

where the second equality follows from (2.13) and (1.2), and for any $d \in \text{Der } \mathfrak{g} \subset \mathfrak{h}(\mathfrak{g})$,

$$\begin{aligned}\mathcal{D}_2(d) &= \mathcal{D}_1(d) - [y - \text{ad}_y, d] \\ &= (x_d - \text{ad}_{x_d}) - (-d(y) - [\text{ad}_y, d]) \\ &= (x_d + d(y)) - (\text{ad}_{x_d} - [\text{ad}_y, d]) \\ &= 0,\end{aligned}\tag{2.23}$$

because $-\text{ad}_{d(y)} = [\text{ad}_y, d] = [d'', d] = \text{ad}_{x_d}$ by (2.20) and (2.21), and $x_d = -d(y)$ (since \mathfrak{g} has zero center). Thus (2.22) and (2.23) show that $\mathcal{D}_2 = 0$ and so \mathcal{D} is inner. This and (2.11) prove that $\mathfrak{h}(\mathfrak{g})$ is complete.

“ \implies ”: Now we prove the necessity. So assume that $\mathfrak{h}(\mathfrak{g})$ is complete. Suppose conversely $C(\text{Der } \mathfrak{g}/\text{ad } \mathfrak{g}) \neq 0$. Then there exists $D \in \text{Der } \mathfrak{g}$ such that

$$D \notin \text{ad } \mathfrak{g} \quad \text{but} \quad [D, \text{Der } \mathfrak{g}] \subset \text{ad } \mathfrak{g}.\tag{2.24}$$

Then for any $d \in \text{Der } \mathfrak{g}$, there exists $x_d \in \mathfrak{g}$ such that $[D, d] = \text{ad}_{x_d}$. Such x_d is unique since $C(\mathfrak{g}) = 0$. Using the fact that $[D, [d, e]] = [[D, d], e] + [d, [D, e]]$, we obtain

$$x_{[d, e]} = -e(x_d) + x_d(e) \quad \text{for} \quad d, e \in \text{Der } \mathfrak{g}.\tag{2.25}$$

We define a linear transformation $\mathcal{D} : \mathfrak{h}(\mathfrak{g}) \rightarrow \mathfrak{h}(\mathfrak{g})$ as follows: $\mathcal{D}|_{\mathfrak{g}} = 0$, and for $d \in \text{Der } \mathfrak{g} \subset \mathfrak{h}(\mathfrak{g})$, we define

$$\mathcal{D}(d) = x_d - \text{ad}_{x_d} \in \mathfrak{g} \rtimes \text{Der } \mathfrak{g} = \mathfrak{h}(\mathfrak{g}).\tag{2.26}$$

From this definition, we have

$$[\mathcal{D}(d), x] = 0 \quad \text{for} \quad x \in \mathfrak{g} \subset \mathfrak{h}(\mathfrak{g}), \quad d \in \text{Der } \mathfrak{g} \subset \mathfrak{h}(\mathfrak{g}).\tag{2.27}$$

Then for any $h = x + d$, $h' = y + e \in \mathfrak{h}(\mathfrak{g})$, by (1.2) and the fact that $\mathcal{D}_{\mathfrak{g}} = 0$, we have

$$\begin{aligned}\mathcal{D}([h, h']) &= \mathcal{D}([d, e]) \\ &= x_{[d, e]} - \text{ad}_{x_{[d, e]}} \\ &= (-e(x_d) + d(x_e)) - ([\text{ad}_{x_d}, e] + [d, \text{ad}_{x_e}]) \\ &= (-e(x_d) - [\text{ad}_{x_d}, e]) + (d(x_e) + [d, \text{ad}_{x_e}]) \\ &= [\mathcal{D}(d), e] + [d, \mathcal{D}(e)] \\ &= [\mathcal{D}(h), h'] + [h, \mathcal{D}(h')].\end{aligned}\tag{2.28}$$

Thus \mathcal{D} is a derivation of $\mathfrak{h}(\mathfrak{g})$. Since $\mathfrak{h}(\mathfrak{g})$ is complete, \mathcal{D} is inner, therefore, there exists $h = y + e \in \mathfrak{h}(\mathfrak{g})$ such that $\mathcal{D} = \text{ad}_h$. For any $x \in \mathfrak{g}$, since $\mathcal{D}|_{\mathfrak{g}} = 0$, we have

$$\begin{aligned} 0 &= \mathcal{D}(x) \\ &= [h, x] \\ &= [y, x] + e(x), \end{aligned} \tag{2.29}$$

i.e., $e = -\text{ad}_y$. Then by (2.26),

$$\begin{aligned} x_d - \text{ad}_{x_d} &= \mathcal{D}(d) \\ &= [h, d] \\ &= -d(y) - [\text{ad}_y, d] \\ &= -d(y) + \text{ad}_{d(y)}. \end{aligned} \tag{2.30}$$

Hence $\text{ad}_{x_d} = -\text{ad}_{d(y)}$. Then for any $d \in \text{Der } \mathfrak{g}$,

$$\begin{aligned} [D, d] &= \text{ad}_{x_d} \\ &= -\text{ad}_{d(y)} \\ &= [\text{ad}_y, d]. \end{aligned} \tag{2.31}$$

Since $\text{Der } \mathfrak{g}$ has zero center, (2.31) implies that $D = \text{ad}_y \in \text{ad}_{\mathfrak{g}}$, a contradiction with (2.24). Thus $C(\text{Der } \mathfrak{g}/\text{ad}_{\mathfrak{g}}) = 0$, and the proof of Theorem 1.1(ii) is complete.

§3. Some Examples

Example 3.1. An $n \times n$ matrix $q = (q_{ij})$ over a field \mathbb{F} of characteristic 0 such that

$$q_{ii} = 1 \quad \text{and} \quad q_{ji} = q_{ij}^{-1}, \tag{3.1}$$

is called a *quantum matrix*. The *quantum torus*

$$\mathbb{F}_q = \mathbb{F}_q[t_1^{\pm}, \dots, t_n^{\pm}], \tag{3.2}$$

determined by a quantum matrix q is defined as the associative algebra over \mathbb{F} with $2n$ generators $t_1^{\pm}, \dots, t_n^{\pm}$, and relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \quad \text{and} \quad t_j t_i = q_{ij} t_i t_j, \tag{3.3}$$

for all $1 \leq i, j \leq n$.

For any $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we denote $t^a = t_1^{a_1} \cdots t_n^{a_n}$. For any $a, b \in \mathbb{Z}^n$, we define

$$\sigma(a, b) = \prod_{1 \leq i, j \leq n} q_{j,i}^{a_j b_i} \quad \text{and} \quad f(a, b) = \prod_{i,j=1}^n q_{j,i}^{a_j b_i}. \tag{3.4}$$

Then we have

$$t^a t^b = \sigma(a, b) t^{a+b}, \quad t^a t^b = f(a, b) t^b t^a \quad \text{and} \quad f(a, b) = \sigma(a, b) \sigma(b, a)^{-1}. \quad (3.5)$$

Note that the commutator of monomials in \mathbb{F}_q satisfies

$$[t^a, t^b] = (\sigma(a, b) - \sigma(b, a)) t^{a+b} = \sigma(b, a) (f(a, b) - 1) t^{a+b}, \quad (3.6)$$

for all $a, b \in \mathbb{Z}^n$. Define the *radical* of f , denoted by $\text{rad}(f)$, by

$$\text{rad}(f) = \{a \in \mathbb{Z}^n \mid f(a, b) = 1 \text{ for all } b \in \mathbb{Z}^n\}. \quad (3.7)$$

It is clear from (3.4) that $\text{rad}(f)$ is a subgroup of \mathbb{Z}^n . The center of \mathbb{F}_q is $C(\mathbb{F}_q) = \sum_{a \in \text{rad}(f)} \mathbb{F} t^a$ and the Lie algebra \mathbb{F}_q has the Lie ideals decomposition

$$\mathbb{F}_q = C(\mathbb{F}_q) \oplus [\mathbb{F}_q, \mathbb{F}_q]. \quad (3.8)$$

By (3.8), we obtain that the Lie algebra $[\mathbb{F}_q, \mathbb{F}_q] \cong \mathbb{F}_q / C(\mathbb{F}_q)$ is perfect and has zero center. From Theorem 1.1, we obtain the following result.

Corollary 3.2. *Let \mathbb{F}_q be a quantum torus as above. Then the derivation algebra $\text{Der}([\mathbb{F}_q, \mathbb{F}_q])$ of $[\mathbb{F}_q, \mathbb{F}_q]$ is complete.*

Example 3.3. A Lie algebra \mathfrak{g} is called a *symmetric self-dual Lie algebra* if \mathfrak{g} is endowed with a nondegenerate invariant symmetric bilinear form B . For any subspace V of \mathfrak{g} , we define $V^\perp = \{x \in \mathfrak{g} \mid B(x, y) = 0, \forall y \in V\}$. Then we can easily check that $[\mathfrak{g}, \mathfrak{g}]^\perp = C(\mathfrak{g})$, where $C(\mathfrak{g})$ is the center of \mathfrak{g} . Then we have

Corollary 3.4. *Assume that \mathfrak{g} is a symmetric self-dual Lie algebra with zero center. Then the Lie algebra $\text{Der } \mathfrak{g}$ is complete.*

Remark 3.5. By [18], we know that the structure of a perfect symmetric self-dual Lie algebra is not clear even in the finite dimension case.

References

- [1] S. Benayadi, “The perfect Lie algebras without center and outer derivations,” *Annals de la Faculte des Sciences de Toulouse* **5** (1996), 203-231.

- [2] R. Carles, “Construction des algebres de Lie completes,” *C. R. Acad. Sci. Paris Ser. I. Math.* **318** (1994), 171-174.
- [3] N. Jacobson, *Lie algebras*, Wiley (Interscience), New York, 1962.
- [4] C. P. Jiang, D. J. Meng and S. Q. Zhang, “Some complete Lie algebras,” *J. Algebras* **186** (1996), 807-817.
- [5] V. G. Kac, *Infinite dimensional Lie algebras*, 3rd ed, Cambridge Univ. Press, 1990.
- [6] G. Leger, “Derivations of Lie algebras III,” *Duke Math. J.* **30** (1963), 637–645.
- [7] D. J. Meng, “Some results on complete Lie algebras,” *Comm. Algebra* **22** (1994), 5457-5507.
- [8] D. J. Meng, “Complete Lie algebras and Heisenberg algebras,” *Comm. Algebra* **22** (1994), 5509-5524.
- [9] D. J. Meng, S. P. Wang, “On the construction of complete Lie algebras,” *J. Algebra* **176** (1995), 621-637.
- [10] D. J. Meng, L. S. Zhu, “Solvable complete Lie algebras I,” *Comm. Algebra* **24** (1996), 4181-4197.
- [11] E. L. Stitzinger, “On Lie algebras with only inner derivations,” *J. Algebra* **105** (1987), 341–343.
- [12] E. V. Schenkman, “A theory of subinvariant Lie algebras,” *Amer. J. Math.* **73** (1951), 453–474.
- [13] S. Tôgô, “Outer derivations of Lie algebras,” *Trans. Amer. Math. Soc.* **128** (1967), 264–276.
- [14] Q. Yang, D. J. Meng, “Derivation algebras of simple Lie algebras,” *Dongbei Shuxue*, In press.
- [15] L. S. Zhu, D. J. Meng, “Derivations of generalized Kac-Moody algebras,” *Chinese Annal of Mathematics* **17A:3** (1996), 271-176 (in chinese).
- [16] L. S. Zhu, D. J. Meng, “The complete Lie algebras with maximal rank nilpotent radicals,” *Chinese Science Bulletin* **44** (1999), 4.
- [17] L. S. Zhu, D. J. Meng, “Complete Lie algebras II,” *Alg. Colloq.* **5** (1998), 289-296.
- [18] L. S. Zhu, D. J. Meng, “A class of Lie algebras with a symmetric invariant non-degenerate bilinear form,” *Acta Math. Sinica* **43** (2000), 1119-1126.